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# Prime Rings Satisfying a Generalized Polynomial Identity

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## 1. INTRODUCTION

The purpose of this paper is to simultaneously generalize recent theorems of Posner and Amitsur, and, in so doing, to replace two proofs, by a single, independent proof. In [1] Posner showed that if  $R$  is a prime ring satisfying a polynomial identity over its centroid then  $R$  can be embedded as both a left and right order in a finite dimensional simple algebra over a field. On the other hand, Amitsur showed in [2] that if  $R$  is a dense ring of linear transformations over a division ring  $D$  (i.e., a primitive ring) which satisfies a generalized polynomial identity over the center  $F$  of  $D$ , then  $R$  contains a transformation of finite rank and  $D$  is finite dimensional over  $F$ . In the present paper we first construct for any prime ring  $R$  a "ring of quotients"  $Q$  whose center  $C$  is a field containing the centroid of  $R$ , and then embed  $R$  in the ring  $S = RC$ . Our main result then states that if  $S$  satisfies a generalized polynomial identity over  $C$ , then  $S$  contains a minimal right ideal  $eS$ ,  $e$  an idempotent of  $S$ , and  $eSe$  is a finite dimensional division algebra over  $C$ .

## 2. THE EXTENDED CENTROID OF A PRIME RING

Let  $R$  be a prime ring, let  $\mathcal{U} = \{U\}$  be the collection of all nonzero twosided ideals of  $R$ , and consider the totality  $T$  of all  $R$ -homomorphisms  $f: U_R \rightarrow R_R$  where  $U$  ranges over  $\mathcal{U}$  and  $U$  and  $R$  are regarded as right  $R$ -modules. We say that  $f$  (acting on  $U$ ) is equivalent to  $g$  (acting on  $V$ ) if  $f = g$  on some  $W \in \mathcal{U}$ , where  $W \subseteq U \cap V$ . This does indeed define an equivalence relation on  $T$ , and we let  $Q$  be the set of all equivalence classes. If  $\hat{f}, \hat{g} \in Q$ , we define  $\hat{f} + \hat{g}$  to be the class determined by  $f + g$  acting on  $U \cap V$ , and we define  $\hat{f}\hat{g}$  to be the class determined by the composite  $f(g)$  acting (from the left) on the ideal  $VU$ . Under these operations it is readily seen that  $Q$  is a ring. All the preceding statements made in this paragraph are easy consequences of the fact that  $\mathcal{U}$  is closed under products and intersections.

The mapping  $a \rightarrow \hat{a}_l$ , where  $a_l$  is the left multiplication  $r \rightarrow ar$ , is clearly a ring homomorphism of  $R$  into  $Q$ . If  $\hat{a}_l = 0$ , then  $aU = 0$  for some  $U \in \mathcal{A}$ , forcing  $a = 0$  since  $R$  is prime. This  $R$  is isomorphically embedded in  $Q$  and, to simplify the notation, we shall consider  $R$  as a subring of  $Q$ .  $Q$  then enjoys the property that for each  $q \in Q$  there exists an ideal  $U \neq 0$  in  $R$  such that  $qU \subseteq R$ .

It is clear that  $Q$  is a prime ring. Indeed, suppose  $qQp = 0$ ,  $0 \neq q \in Q$ ,  $0 \neq p \in Q$ . There exist ideals  $U \neq 0$  and  $V \neq 0$  of  $R$  such that  $qU \subseteq R$  and  $pV \subseteq R$ . Since  $q \neq 0$  and  $p \neq 0$ , there exist  $u \in U$  and  $v \in V$  such that  $qu \neq 0$  and  $p v \neq 0$ . Thus  $(qu)R(pv) \subseteq qQpv = 0$ , a contradiction to the primeness of  $R$ .

The center  $C$  of  $Q$  is a commutative integral domain containing no zero divisors in  $Q$ . Let  $0 \neq c \in C$  and choose a nonzero ideal  $U$  of  $R$  such that  $cU \subseteq R$ .  $cU$  is itself a nonzero ideal of  $R$  and the mapping  $cu \rightarrow u$  of  $cU$  into  $R$  induces an element  $d$  of  $Q$ . Thus  $dcu = u$  for all  $u \in U$ , and so  $dc = 1$ , which shows that  $C$  is a field.

We now let  $S = RC$ , a subring of  $Q$  containing  $R$ . We shall call  $C$  the extended centroid of  $R$  and  $S$  the central closure of  $R$ . If  $1 \in R$ , then  $C$  is the center of  $S$ . The same proof used in showing that  $Q$  was prime may be employed to show that  $S$  is prime. We remark in passing that if no nonzero element of  $R$  annihilates a so-called dense right ideal of  $R$  (see [3], Chapter 4, Section 3), we could have followed our procedure in [4] and first formed Utumi's complete ring of right quotients  $Q'$  or  $R$ . The center  $C'$  of  $Q'$  would then have been isomorphic to  $C$ .

Our first theorem is a slight generalization of [4], Theorem 4, which in turn was originally inspired by [1], p. 215, Lemma 6(a). For completeness we include the proof.

**THEOREM 1.** *Let  $a, b \in S$  such that  $axb = bxa$  for all  $x \in R$  (hence for all  $x \in S$ ). Then  $a$  and  $b$  are  $C$ -dependent.*

*Proof.* We may assume that  $a \neq 0$  and  $b \neq 0$ . Let  $U$  be a nonzero ideal of  $R$  such that  $aU \subseteq R$  and  $bU \subseteq R$ , and set  $V = UaU$ . We define a mapping  $f: V \rightarrow R$  according to the rule

$$\sum_i x_i a y_i \rightarrow \sum_i x_i b y_i, \quad x_i, y_i \in U.$$

Suppose  $\sum_i x_i a y_i = 0$ . Then

$$0 = br \sum_i x_i a y_i = \sum_i b(r x_i) a y_i = \sum_i a(r x_i) b y_i = ar \sum_i x_i b y_i.$$

Thus  $(aU)R(\sum_i x_i b y_i) = 0$  and so  $\sum_i x_i b y_i = 0$  since  $R$  is prime. This shows

that  $f$  is well-defined,  $f$  is an  $R$ -homomorphism because  $f\{(xay)r\} = xbyr = f(xay)r$  for all  $x, y \in U$  and  $r \in R$ . Let  $q$  denote the element of  $Q$  determined by  $f$  and let  $p$  be any element of  $Q$ , with  $pW \subseteq R$  for some nonzero ideal  $W$  of  $R$ . For  $x, y \in U$  and  $w \in W$  we have  $(qp)(wxay) = q\{(pw)xay\} = (pw)xby = p\{wxby\} = pq(wxay)$ , showing that  $(qp - pq)WV = 0$ . Thus  $qp = pq$  for all  $p \in Q$ , and so  $q \in C$ . In particular,  $x(qa - b)y = q xay - xby = 0$  for all  $x, y \in U$ , yielding  $V(qa - b)V = 0$ . Since  $R$  is prime, we obtain  $qa = b$ .

**THEOREM 2.** *Let  $a_1, a_2, \dots, a_m$  be  $C$ -independent elements of  $S$  and let  $b_1, b_2, \dots, b_m \in S$ , with  $b_1 \neq 0$ . Suppose  $B = \{\sum_{i=1}^m a_i x b_i \mid x \in S\}$  is finite dimensional over  $C$ . Then*

- (a)  $B \neq 0$ ,
- (b)  $S$  has a minimal right ideal  $eS$ ,
- (c)  $eSe$  is a finite dimensional division algebra over  $C$ .

*Proof.* The proof is by induction on  $m$ . For  $m = 1$  we have  $B = aSb$ , and so  $B \neq 0$  results from the primeness of  $S$ . Thus there exists a  $C$ -basis  $v_1, v_2, \dots, v_k$  of  $B$ ,  $k \geq 1$ , so that  $axb = \sum_{i=1}^k \lambda_i(x) v_i$  for all  $x \in S$ , where  $\lambda_i(x) \in C$ . Choose  $r \in S$  such that  $bra \neq 0$ , and set  $d = br$ . Then  $axd = \sum_{i=1}^k \lambda_i(x)(v_i r)$  for all  $x \in S$  and so  $aSd$  is an at most  $k$ -dimensional algebra over  $C$ . Since  $da \neq 0$  and  $S$  is prime,  $aSd$  properly contains its (nilpotent) radical  $N$ .  $aSd/N$  is a finite dimensional semi-simple algebra and in particular has an identity element  $\bar{u}$ . From this it is well known that  $aSd$  itself contains a nonzero idempotent  $f$ , since  $N$  is nilpotent. Then  $fSf$  ( $\subseteq aSd$ ) is a finite dimensional prime algebra over  $C$  and so  $fSf \cong D_n$ , where  $D$  is a finite dimensional division algebra over  $C$ . Choose  $e$  to be a (primitive) idempotent of  $fSf$ , so that  $eSe = e(fSf)e \cong D$ . Thus  $eSe$  is a finite dimensional division algebra over  $C$ , which in turn implies that  $eS$  is a minimal right ideal of  $S$ .

Next suppose that  $\sum_{i=1}^m a_i x b_i = \sum_{j=1}^k \lambda_j(x) v_j$  for all  $x \in S$ , where  $m > 1$ ,  $\{a_i\}$  independent,  $b_1 \neq 0$ ,  $\{v_j\}$  basis for  $M$ ,  $\lambda_j(x) \in C$ . If  $b_i = \gamma_i b_1$ ,  $\gamma_i \in C$ ,  $i = 2, 3, \dots, m$ , then we have  $axb_1 = \sum_{j=1}^k \lambda_j(x) v_j$  for all  $x \in S$ , where  $a = a_1 + \sum_{i=2}^m \gamma_i a_i \neq 0$ . This case has already been worked out. Hence, by reordering subscripts, we may assume that  $b_1$  and  $b_2$  are  $C$ -independent. Multiplication on the right by  $tb_1$ , where  $t \in S$ , yields

$$\sum_{i=1}^m a_i x b_i t b_1 = \sum_{j=1}^k \lambda_j(x) v_j t b_1 \quad (1)$$

for all  $x, t \in S$ . On the other hand

$$\sum_{i=1}^m a_i (x b_1 t) b_i = \sum_{j=1}^k \lambda_j(x b_1 t) v_j \quad (2)$$

for all  $x, t \in S$ . Subtracting (2) from (1), we obtain

$$\sum_{i=2}^m a_i x(b_i t b_1 - b_1 t b_i) = \sum_{j=1}^k \{\lambda_j(x) v_j t b_1 - \lambda_j(x b_1 t) v_j\} \quad (3)$$

for all  $x, t \in S$ . By Theorem 1 there exists  $t_0 \in S$  such that  $b_2 t_0 b_1 - b_1 t_0 b_2 \neq 0$ . Setting  $b'_i = b_i t_0 b_1 - b_1 t_0 b_i$ ,  $w_j = v_j t_0 b_1$ , and  $\mu_j(x) = -\lambda_j(x b_1 t_0)$  we then have  $\sum_{i=2}^m a_i x b'_i = \sum_{j=1}^k \{\lambda_j(x) w_j + \mu_j(x) v_j\}$ , with  $b'_2 \neq 0$ . By induction the proof is now complete.

### 3. GENERALIZED POLYNOMIAL IDENTITIES

Let  $R$  be a prime ring and let  $S = RC$  be the central closure of  $R$ . Following Amitsur, we form the so-called  $C$ -universal product  $S\langle x \rangle$  of the  $C$ -algebra  $S$  and the free  $C$ -algebra  $C[x_1, x_2, \dots, x_n, \dots]$  in noncommuting indeterminates  $x_1, x_2, \dots, x_n, \dots$ . Roughly speaking, the elements of  $S\langle x \rangle$  are of the form

$$f = \sum \beta_k a_{i_0} x_{j_1} a_{i_1} \cdots a_{i_{n-1}} x_{j_n} a_{i_n}$$

where  $\beta_k \in C$ ,  $a_{i_k} \in S$ . For more precise details concerning the construction and relevant properties of  $S\langle x \rangle$  we refer the reader to [2], Section 4.

**DEFINITION.**  $S$  is said to satisfy a nontrivial generalized polynomial identity over  $C$  ( $S$  is G.P.I.) if there is a nonzero element  $f(x_1, x_2, \dots, x_n)$  in  $S\langle x \rangle$  such that  $f(x_1, s_2, \dots, s_n) = 0$  for all  $s_1, s_2, \dots, s_n \in S$ .

The degree of the monomial  $a_0 x_{i_1} a_1 x_{i_2} \cdots a_{n-1} x_{i_n} a_n$  (all  $a_i$ 's  $\neq 0$ ) is  $n$ , and the degree of an element  $f$  of  $S\langle x \rangle$  is the maximum degree of its monomials (assuming that a representation of  $f$  as a sum of monomials is chosen so that the degree of the monomial of highest degree is minimal). If  $S$  satisfies a generalized polynomial identity of degree  $n$ ,  $n$  minimal, then the usual linearization process may be used to obtain a nontrivial generalized (homogeneous) multilinear identity of degree  $n$  in  $x_1, x_2, \dots, x_n$ :

$$\sum \beta_i a_{i_0} x_{j_1} a_{i_1} \cdots a_{i_{n-1}} x_{j_n} a_{i_n} = 0$$

where each monomial is of the same fixed degree  $n$ . We are now in a position to prove the main theorem of our paper.

**THEOREM 3.** *Let  $R$  be a prime ring and let  $S = RC$  be the central closure of  $R$ . Then  $S$  satisfies a generalized polynomial identity over  $C$  if and only if  $S$*

contains a minimal right ideal  $eS$  (hence  $S$  is primitive) and  $eSe$  is a finite dimensional division algebra over  $C$ .

*Proof.* If  $S$  enjoys the latter two properties, then we first note that  $eSe$  satisfies the standard identity  $s(x_{i_1}, x_{i_2}, \dots, x_{i_n}) = \sum_i \pm x_{i_1} x_{i_2} \cdots x_{i_n} = 0$  if  $n$  exceeds the dimension of  $eSe$  over  $C$ . In other words  $S$  itself satisfies the generalized identity

$$\sum ex_{i_1} ex_{i_2} e \cdots ex_{i_n} e = 0.$$

Conversely, suppose  $S$  satisfies a nontrivial generalized polynomial identity of minimal degree  $n$ . Without loss of generality we may assume this identity is homogeneous multilinear of degree  $n$  so that it has the form:

$$f(x_1, x_2, \dots, x_n) = \sum_{i=1}^m a_i x_1 f_i(x_2, \dots, x_n) + g(x_1, x_2, \dots, x_n) = 0$$

where  $a_1, a_2, \dots, a_m$  are  $C$ -independent elements of  $S$ ,  $f_i$  are nonzero generalized homogeneous multilinear polynomials of degree  $n - 1$ , and  $g$  is a sum of monomials none of which have  $x_1$  as their first variable. If  $x_1$  appears nontrivially as the last variable in some monomial of  $g$  then we may further break up the identity so as to look like

$$f = \sum_{i=1}^m a_i x_1 f_i + \sum_{i=1}^k g_i x_1 b_i + \sum p_i x_1 q_i = 0 \quad (1)$$

where  $b_1, b_2, \dots, b_k$  are  $C$ -independent elements of  $S$ ,  $g_i$  is of degree  $n - 1$ , and  $p_i$  and  $q_i$  are generalized polynomials of positive degree. For  $t \in S$ , multiplication of (1) on the right by  $tb_1$  yields

$$\sum_{i=1}^m a_i s_1 f_i t b_1 + \sum_{i=1}^k g_i s_1 b_i t b_1 + \sum p_i s_1 q_i t b_1 = 0 \quad (2)$$

for all  $s_1, s_2, \dots, s_n, t \in S$ . Substitution of  $x_1$  by  $s_1 b_1 t$  in (1) leads to

$$\sum_{i=1}^m a_i s_1 b_1 t f_i + \sum_{i=1}^k g_i s_1 b_1 t b_i + \sum p_i s_1 b_1 t q_i = 0 \quad (3)$$

for all  $s_1, s_2, \dots, s_n, t \in S$ . Subtraction of (3) from (2) gives

$$\sum_{i=1}^m a_i s_1 (f_i t b_1 - b_1 t f_i) + \sum_{i=2}^k g_i s_1 (b_i t b_1 - b_1 t b_i) + \sum p_i s_1 (q_i t b_1 - b_1 t q_i) = 0 \quad (4)$$

for all  $s_1, s_2, \dots, s_n, t \in S$ .

Suppose  $f_1 t b_1 - b_1 t f_1 = 0$  for all  $s_2, \dots, s_n, t \in S$ . By Theorem 1,  $f_1(s_2, \dots, s_n) = \lambda(s_2, \dots, s_n) b_1$  for all  $s_2, \dots, s_n \in S$ , where  $\lambda(s_2, \dots, s_n) \in C$ . By the minimality of  $n$ ,  $f_1(r_2, r_3, \dots, r_n) \neq 0$  for some  $r_2, r_3, r_n \in S$ . Define  $h(x_2) = f_1(x_2, r_3, \dots, r_n)$ , and note that  $h(x_2) \neq 0$  in  $S\langle x \rangle$  since  $h(r_2) \neq 0$ .  $h(x_2)$  may be written  $h(x_2) = \sum_{i=1}^j c_i x_2 d_i$ , where  $\{c_i\}$  are  $C$ -independent and the  $d_i$  are nonzero elements of  $S$ . Thus  $h(x) = \mu(x) b_1$  for all  $x \in S$ , where  $\mu(x) = \lambda(x, r_3, \dots, r_n) \in C$ , and so by Theorem 2 we are finished.

We may therefore assume that  $f_1 t_0 b_1 - b_1 t_0 f_1 \neq 0$  for some  $r_2, r_3, \dots, r_n, t_0 \in S$ . Setting

$$f'_i = f_i t_0 b_1 - b_1 t_0 f_i, \quad b'_i = b_i t_0 b_1 - b_1 t_0 b_i, \quad \text{and} \quad q'_i = q_i t_0 b_1 - b_1 t_0 q_i$$

we have, in view of (4), that  $S$  satisfies

$$\sum_{i=1}^m a_i x_1 f'_i + \sum_{i=2}^k g'_i x_1 b_i + \sum p_i x_1 q'_i = 0 \quad (5)$$

where  $f'_1(r_2, \dots, r_n) \neq 0$ . (5) is not a trivial identity, since this would imply that  $\sum_{i=1}^m a_i x_1 f'_i$  were trivial, which in turn would contradict Theorem 2 by specializing  $x_i = r_i, i = 2, 3, \dots, n$ . Furthermore, we make the important observation that in transforming the identity (1) to the identity (4) in no monomial has the order in which the variables  $x_1, x_2, \dots, x_n$  appear been changed (some monomials may have disappeared).

Repetition of the above process at most  $k$  times will enable us to transform our original identity (1) into a nontrivial one of the form

$$\sum_{i=1}^m a_i x_1 f_i(x_2, \dots, x_n) + g(x_1, x_2, \dots, x_n) = 0 \quad (6)$$

in which  $x_1$  never appears as the last variable in any monomial of  $g$ , and in which no new order of the variables is introduced in any monomial.

We may assume that  $x_1, x_2, \dots, x_r, r \leq n$ , are those variables which appeared first in some monomial of the original identity. Applying the preceding process to each of these variables, we may in a finite number of steps transform the original identity to a nontrivial one of the form

$$\sum a_i x_1 f_i + \sum b_i x_2 g_i + \dots + \sum d_i x_r h_i = 0 \quad (7)$$

in which  $\{a_i\}, \{b_i\}, \dots, \{d_i\}$  are  $C$ -independent sets in  $S$ , and  $f_i, g_i, \dots, h_i$  are nonzero  $n-1$  degree generalized polynomials in which none of  $x_1, x_2, \dots, x_r$  ever appear as the last variable in any monomial. Since some variable must

appear last in each monomial, we conclude that  $r < n$ . By the minimality of  $n$ ,  $f_1(r_2, r_3, \dots, r_n) \neq 0$  for some  $r_2, r_3, \dots, r_n \in S$ . Let

$$\begin{aligned} f'_i(x_2, \dots, x_{n-1}) &= f_i(x_2, x_3, \dots, x_{n-1}, r_n), \\ g'_i(x_1, x_3, \dots, x_{n-1}) &= g_i(x_1, x_3, \dots, x_{n-1}, r_n), \dots, h'_i(x_1, x_2, \dots, x_{r-1}, x_{r+1}, \dots, x_{n-1}) \\ &= h_i(x_1, \dots, x_{r-1}, x_{r+1}, x_{n-1}, r_n). \end{aligned}$$

We claim that

$$\sum a_i x_1 f'_i + \sum b_i x_2 g'_i + \dots + \sum d_i x_r h'_i = 0 \quad (8)$$

is a nontrivial identity of degree  $n - 1$ . If (8) is trivial, then it would follow that  $\sum a_i x_1 f'_i$  is trivial. Setting  $c_i = f'_i(r_2, r_3, \dots, r_n)$  we would then have  $\sum a_i x c_i = 0$  for all  $x \in S$ , with  $\{a_i\}$   $C$ -independent and  $c_1 = f(r_2, r_3, \dots, r_n) \neq 0$ . This is a contradiction to Theorem 2, and therefore (8) must be a nontrivial identity. This, however, now contradicts the minimality of  $n$ .

#### 4. COROLLARIES

We first specialize to the situation where  $R$  is a primitive ring.  $R$  may be considered as an irreducible ring of endomorphisms of an additive abelian group  $V$ , so that  $D = \text{Hom}_R(V, V)$  is a division ring. Let  $F$  be the center of  $D$ , and set  $T = RF$ , a subring of  $\text{Hom}(V, V)$  with same division ring  $D$ . Clearly  $F$  is contained in the extended center  $C$  of  $T$ , and, conversely, the proof of [4], Theorem 12, shows that  $C$  is contained in  $F$ . Thus  $C = F$  and so the central closure of the ring  $T$  is  $T$  itself.

**THEOREM 4 (Kaplansky).** *Let  $R$  be a primitive ring satisfying a polynomial identity over its centroid. Then  $R$  is a finite dimensional central simple algebra.*

*Proof.* We may assume that  $R$  satisfies a homogeneous multilinear identity over its centroid  $Z$ :

$$x_1 x_2 \cdots x_n + \sum_{\alpha \neq I} \alpha_i x_{i_1} x_{i_2} \cdots x_{i_n} = 0, \quad \alpha_i \in Z, \quad (1)$$

which is also satisfied by  $T$ . If  $[V : D] \geq n$ , then  $T$  contains a subring which has as a homomorphic image  $D_n$ , the  $n \times n$  matrices over  $D$ . (1) is therefore satisfied by  $D_n$ , but this is clearly impossible if we set  $x_1 = e_{11}$ ,  $x_2 = e_{12}$ ,  $x_3 = e_{22}$ , etc., where the  $e_{ij}$  are the usual matrix units of  $D_n$ . Hence  $V$  is finite dimensional over  $D$ . Finally, by Theorem 3,  $D$  is finite dimensional over its center  $F$ .

The primitive rings  $R$  studied by Amitsur in [2] are essentially those which are  $F$ -algebras (i.e.,  $RF \subseteq R$ ), and so our Theorem 3 directly implies

**THEOREM 5 (Amitsur).** *Let  $R$  be a primitive ring such that  $RF \subseteq R$ , where  $F$  is the center of the associated division ring  $D$ . Then  $R$  satisfies a generalized polynomial identity over  $F$  if and only if  $R$  contains a minimal right ideal and  $D$  is finite dimensional over  $F$ .*

The other main corollary of our theorem is

**THEOREM 6 (Posner).** *Let  $R$  be a prime ring satisfying a polynomial identity over its centroid  $Z$ . Then  $R$  can be embedded as either a left or right order in its central closure  $S = RC$ , and  $S$  is a finite dimensional central simple algebra over  $C$ .*

*Proof.* We may first assume that  $R$  satisfies a homogeneous multilinear identity

$$\sum \alpha_i x_{i_1} x_{i_2} \cdots x_{i_n} = 0, \quad \alpha_i \in Z, \quad (2)$$

and that  $S = RC$  satisfies this same identity. Furthermore, because different monomials have the variables in a different order (2) remains a nontrivial identity over  $C$ . By Theorem 3,  $S$  is in particular primitive and so, by Kaplansky's Theorem,  $S$  is a finite dimensional central simple algebra over  $C$ .

In order to show that  $R$  is an order in  $S$  we shall first show that every nonzero ideal  $U$  of  $R$  contains a regular element. Indeed, write  $S = D_k$  and let  $e_1, e_2, \dots, e_k$  be the usual orthogonal idempotents in  $S$ . Write  $e_i = \sum_j r_{ij} c_{ij}$ ,  $r_{ij} \in R$ ,  $c_{ij} \in C$ . Since there are only a finite number of  $c_{ij}$ , there exists a nonzero ideal  $W$  of  $R$  such that  $W \subseteq U$  and  $c_{ij}W \subseteq R$  for all  $i, j$ . It is easy to see then that  $e_i W^3 e_i \subseteq U$ . Furthermore,  $e_i W^3$  is a nonzero right ideal of  $R$ , and, since  $R$  is prime,  $e_i W^3 e_i \neq 0$ . We now select  $0 \neq u_i \in e_i W^3 e_i \subseteq U$ ,  $i = 1, 2, \dots, k$ , and set  $u = u_1 + u_2 + \cdots + u_k$ . In  $S$ ,  $u$  clearly has rank  $k$  and so must be a regular element of  $R$ .

Now, in order to show that every element of  $S$  can be written in the form  $ab^{-1}$ ,  $a, b \in R$ ,  $b$  regular, it suffices, since  $S = RC$ , to show that for any finite set of elements  $c_1, c_2, \dots, c_m \in C$  we can find  $a_1, a_2, \dots, a_m, b \in R$ ,  $b$  regular, such that  $c_i = a_i b^{-1}$ ,  $i = 1, 2, \dots, m$ . Certainly there is a nonzero ideal  $U$  of  $R$  such that  $c_i U \subseteq R$ ,  $i = 1, 2, \dots, m$ . From the preceding paragraph  $U$  contains a regular element  $b$ , and hence  $c_i b = a_i \in R$ ,  $i = 1, 2, \dots, m$ , or in  $S$ ,  $c_i = a_i b^{-1}$ ,  $i = 1, 2, \dots, m$ . This completes the proof of Posner's Theorem.

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